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LETTER TO THE EDITOR

New relations between the monomer–dimer and the Yang–Lee models

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Abstract. A Hamiltonian approach to the monomer–dimer model is derived by a special truncation of the spin operators in the Yang–Lee model. At its critical point, the negative fugacity dimer problem is proved to belong to the universality class of the Yang–Lee singularity as previously conjectured.

Recently there has been a new interest in critical behaviour in the presence of a purely imaginary symmetry-breaking field (see, for example, Fisher (1980) and references therein). The singularity corresponds to an accumulation of zeros of the partition function for $T > T_c$ and near a critical value of the field, the so-called Yang–Lee (YL) edge (Yang and Lee 1952).

In addition to new analytical results (Bessis *et al* 1976, Baker *et al* 1979, Fisher 1980) and new accuracy in determining the critical exponent (Kurtze and Fisher 1979, de Alcantara Bonfim *et al* 1980, Jullien *et al* 1981, Uzelac and Jullien 1981), relations to the behaviour of other systems near criticality have been also obtained. These include branched dilute polymers (animals) in $d+2$ dimensions (Parisi and Sourlas 1981) and electronic localisation in a random potential near $d = 8$ (Lubensky and McKane 1981). For the localisation problem, the behaviour near the mobility edge has been more explicitly related to the YL edge singularity in the one-dimensional case (Kapitulnik and Shapir unpublished).

A few years ago, another model was conjectured to share the same critical behaviour, namely the monomer–dimer (MD) model with negative dimer fugacity (Kurtze and Fisher 1979, Baker and Moussa 1978). This conjecture implicitly assumes that the YL model is not driven away from its universality class in the extreme high-temperature limit. Based on this assumption, a calculation of the YL exponent was performed using the MD fugacity expansion (Kurtze and Fisher 1979, Gaunt 1969), achieving good agreement with the results obtained by other methods (Kortman and Griffiths 1971, Fisher 1978).

The purpose of the present communication is to relate these two models in their Hamiltonian formulations and to show that indeed their critical behaviours are described by the same field theory, at least near the upper critical dimensionality $d_c = 6$.

We start by deriving the Hamiltonian formulation of the MD model from a truncated version of the spin model. The YL Ising model is represented by the following reduced Hamiltonian:

$$-\beta\mathcal{H} = K \sum_{\langle ij\rangle} S_i S_j + \mathrm{i}H \sum_i S_i, \qquad S_j = \pm 1, \tag{1}$$

and the sum is over nearest-neighbour sites on a d-dimensional lattice. For $K < K_c$ at the critical value of the field $H_c(K)$, the density of zeros $g(H)$ has a branch cut singularity (Yang and Lee 1952),

$$g(H) \approx [H - H_c(K)]^\sigma, \qquad H \geq H_c, \tag{2}$$

and $g(H)$ vanishes for $0 < H < H_c$.

This being a single scaling-field theory, all other exponents are related to σ (Fisher 1978). It is plausible to assume that $H_c(K)$ is a monotonic non-increasing function of K. This is confirmed by exact solutions for $d = 1$ (Yang and Lee 1952), $d = \infty$ (Baker and Moussa 1978) and from series expansions for $d = 2$ and $d = 3$ (Kortman and Griffiths 1971). Near the critical point $K = K_c$, $H_c = 0$, this follows from scaling arguments which give the actual dependence of H on K in the critical range (Suzuki 1976). In the high-temperature limit $K \to 0$, H_c approaches $\pi/2$ from below. If this limit is taken with $z = \tanh K\, tg^2 H$ remaining fixed, only the shortest 'strings' survive in the diagrammatic expansion (Kurtze and Fisher 1979). The partition function is, in that limit, equivalent to the generating function $G(-z)$ for the number of configurations (n_k) for k hard rods (dimers) with negative fugacity $-z$:

$$G(-z) = \sum_k n_k(-z)^k. \tag{3}$$

In order to generate the MD Hamiltonian from the spin model, we first choose to approach the YL critical line $H_c(K)$ along the straight line $H = cK$ (c-coordination number). Due to their respective monotonicity properties the two lines cross at a single point. Along this line the Hamiltonian (1) reads

$$-\beta\mathcal{H} = -K \sum_{\langle ij\rangle} \tau_i \tau_j \tag{4}$$

in terms of the new variables $\tau_j = 1 - \mathrm{i}S_j$.

We next define operators a_j by a 'truncated' version of τ_j with the following trace properties:

$$\mathrm{Tr}\, a_j = \mathrm{Tr}\, \tau_j = 2, \qquad \mathrm{Tr}\, a_j^2 = \mathrm{Tr}\, \tau_j^2 = 0, \tag{5a, b}$$

$$\mathrm{Tr}\, a_j^k = 0 \qquad \text{for } k \geq 3. \tag{5c}$$

The last requirement (5c) is a constraint not fulfilled by the τ_j. It is easy to see that the Hamiltonian (4) written in terms of the a_j is the required MD Hamiltonian. By expanding the partition function,

$$Z(z) = 2^{-N} \operatorname*{Tr}_{\{a\}} \exp\Big(-z \sum_{\langle ij\rangle} a_i a_j\Big) = 2^{-N} \operatorname*{Tr}_{\{a\}} \prod_{\langle ij\rangle} (1 - z a_i a_j + \ldots), \tag{6}$$

and using (5) we obtain the generating function (3). This function may also be expressed in terms of bilinear forms of anticommuting (Grassmann) variables with the appropriate measure (Samuel 1980).

From our simple Hamiltonian approach the field theory of the MD model can be constructed using the identity

$$\exp\Big(-\frac{z}{2} \sum_{(ij)} a_i A_{ij} a_j\Big) = (\det A_{ij})^{-1/2} z^{-N/2} \int \Big[\prod_i \mathrm{d}\phi_i\Big] \exp\Big(-\frac{1}{2z} \sum_{(ij)} \phi_i A_{ij}^{-1} \phi_j + \mathrm{i} \sum_i \phi_i a_i\Big) \tag{7}$$

where $A_{ij}=1$ if i and j are nearest neighbours and vanishes otherwise and N is the number of sites. The trace over the a's may be performed, giving

$$G(-z)=(\det A_{ij})^{-1/2}z^{-N/2}\int\left[\prod_i d\phi_i\right]\exp\left(-\frac{1}{2z}\sum_{(ij)}\phi_i A_{ij}^{-1}\phi_j+\sum_i \ln[(1+i\phi_i)]\right). \tag{8}$$

Neglecting fluctuations, $G(-z)$ is proportional to $\exp[-N\Gamma_0(\bar{\phi})]$. The zero-loop approximation of the potential in the thermodynamic limit is (up to a constant)

$$\Gamma_0(\bar{\phi})=-(1/2cz)\bar{\phi}^2+\ln(1+i\bar{\phi}), \tag{9}$$

where

$$\bar{\phi}=\tfrac{1}{2}i[1-(1-4cz)^{1/2}] \tag{10}$$

is the solution of the saddle-point equation

$$\left.\frac{\partial\Gamma}{\partial\phi}\right|_{\phi=\bar{\phi}}=-\frac{\bar{\phi}}{zc}+\frac{i}{(1+i\bar{\phi})}=0. \tag{11}$$

By substituting $\phi=\bar{\phi}+\psi$ and expanding A_{ij}^{-1} in terms of gradients one obtains the effective Lagrangian for this model:

$$L[\psi]=\tfrac{1}{2}r\psi^2+\tfrac{1}{2}(\nabla\psi)^2+\frac{g}{3!}\psi^3+\frac{u}{4!}\psi^4+\ldots. \tag{12}$$

The quartic term has the correct stability sign and higher terms are irrelevant. The mean-field critical point is at

$$r\approx[1/zc+(\bar{\phi}/zc)^2]=0 \tag{13a}$$

or

$$z_{co}=1/4c \tag{13b}$$

(and so the transition is absent for positive fugacity). The ψ^3 interaction has an imaginary coupling-constant,

$$g\approx(\bar{\phi}/zc)^3, \tag{14}$$

reproducing the Yang–Lee field theory (Fisher 1978).

We have thus proved that both models are in the same universality class and have the same critical properties near the upper critical dimensionality $d_c=6$. The situation at lower dimensionalities depends crucially on the anomalous dimensions of other operators near the non-trivial fixed point. A recent investigation (Fucito and Parisi 1981) for another (tensorial) ϕ^3 theory, namely the q-state Potts model, shows that other operators may become relevant at lower dimensionalities. If it is the case for the present field theory, further analyses have to be performed to answer the question of whether these two models share the same asymptotic singular behaviour in any dimensionality.

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Note added in proof. In $4-\varepsilon$ dimensions the irrelevance of the most dangerous operators near the fixed point was shown (Kirkham and Wallace 1979).

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